Partial Teleportation of Entanglement in the Noisy Environment

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(February 1, 2008)

Partial teleportation of entanglement is to teleport one particle of an entangled pair through a quantum channel. This is conceptually equivalent to quantum swapping. We consider the partial teleportation of entanglement in the noisy environment, employing the Werner-state representation of the noisy channel for the simplicity of calculation. To have the insight of the many-body teleportation, we introduce the measure of correlation information and study the transfer of the correlation information and entanglement. We find that the fidelity gets smaller as the initial-state is entangled more for a given entanglement of the quantum channel. The entangled channel transfers at least some of the entanglement to the final state.

I. INTRODUCTION

Quantum teleportation of a single-particle state has been extensively studied both theoretically [1] and experimentally [2]. Quantum teleportation reproduces an unknown quantum state at a remote place while the original state is destroyed [1]. The key of the quantum teleportation is the quantum channel composed of the quantum entangled pair. If the quantum channel is maximally entangled, for example, by using the singlet state, the quantum state is perfectly reproduced at the remote place and the fidelity of the teleportation is unity. However, in the real world, the quantum channel lies in the noisy environment, which degrades the entanglement of the channel. The less is the quantum channel entangled, the smaller is the fidelity [3–5]. It has also been found that the fidelity of the quantum teleportation is always larger than that of any classical communication protocol even in the noisy environment [3].

In this paper, we are interested in partial teleportation of an entangled state of a two spin-1/2 system. An entangled pair of particles are prepared by Alice who wants to teleport one of the entangled pair to Bob as shown in Fig. 1. If the quantum channel is maximally entangled, the partial teleportation is nothing more than entanglement swapping [6,7]. Bennett *et al.* [1] argued that teleportation is a linear operation for the perfect quantum channel and could be extended to what is now called entanglement swapping [6], which has been experimentally realised [7]. Entanglement swapping was considered for a more generalised multi-particle system [8] and for concentration of partially entangled states [9]. In this paper, we analyse the environmental effects on the partial teleportation of the entangled state, considering the entanglement transfer and the fidelity. We define a measure of entanglement using the partial transposition. Although this measure does not completely agree with the entropy of entanglement for a pure state, it is useful and gives qualitative information. As it is easily calculated and satisfies important conditions of the measure of entanglement, we use it to analyse the partial teleportation in this paper. The concept of correlation information is also introduced. The correlation information we define is, in general, dependent on classical and quantum correlation but we show that it bears a simple and useful linear relation for the partial teleportation.

We assume that the initial entangled state is pure and the mixed quantum channel is represented by the Werner state. We show that the calculation is much simpler as we employ the Werner-state channel while we do not lose the generality of the treatment. Entanglement transfer was extensively studied by Schumacher [10] and teleportation through general channels was considered by Horodecki et al. [11].

II. MEASURES OF ENTANGLEMENT

For a pure state $\hat{\rho}$ of a bipartite system, one can choose the entropy S of entanglement as a measure of entanglement, where

$$S = \mathrm{Tr}\hat{\rho}_a \log_2 \hat{\rho}_a = \mathrm{Tr}\hat{\rho}_b \log_2 \hat{\rho}_b \tag{1}$$

where $\hat{\rho}_{a,b} = \mathrm{Tr}_{b,a}\hat{\rho}$ is the reduced density matrix for the subsystem a or b. For a mixed state, there have been many definitions for the measure of entanglement such as entanglement of formation [12], quantum relative entropy, and Bures metric [13,14]. Every measure $E(\hat{\rho})$ should satisfy the following necessary conditions for a given density matrix $\hat{\rho}$ [13,14],

(C.1) $E(\hat{\rho}) = 0$ if and only if $\hat{\rho}$ is separable.

(C.2) A local unitary transformation leaves $E(\hat{\rho})$ invariant;

$$E(\hat{U}_1 \otimes \hat{U}_2 \hat{\rho} \hat{U}_1^\dagger \otimes \hat{U}_2^\dagger) = E(\hat{\rho}) \tag{2}$$

for all unitary operators $\hat{U}_1$ and $\hat{U}_2$.

(C.3) $E(\hat{\rho})$ cannot increase under local general measurements (LGM), classical communications (CC), and post selection of subensemble (PSS),

$$\sum p_i E(\hat{\rho}_i) \leq E(\hat{\rho}), \tag{3}$$

where $p_i \hat{\rho}_i = \hat{A}_i \otimes \hat{B}_i \hat{\rho} \hat{A}_i^\dagger \otimes \hat{B}_i^\dagger$ with $p_i = \mathrm{Tr} \hat{A}_i \otimes \hat{B}_i \hat{\rho} \hat{A}_i^\dagger \otimes \hat{B}_i^\dagger$; two set of LGM operators $\{\hat{A}_i\}$ and $\{\hat{B}_i\}$ are classically correlated by CC.

The requirement of the condition (C.1) is clear since a separable state is just classically correlated and should be independent of the entanglement. Since a local unitary operation is performed only locally, it cannot affect any entanglement, required by the condition (C.2). The condition (C.3) is related with the purification procedure which selects a subensemble of maximally-quantum-correlated pairs among an impure ensemble [12]. The purification procedure can distill maximally entangled states such that $E(\hat{\rho}_i) \geq E(\hat{\rho})$ for a certain route i, but the average entanglement cannot increase over the whole ensemble since quantum nonlocal operations are not introduced, represented in the condition (C.3).

For a two spin-1/2 system, we define the measure of entanglement in terms of the negative eigenvalues of the partial transposition of the state. Consider a density matrix $\hat{\rho}$ for a two spin-1/2 system and its partial transposition $\hat{\sigma} = \hat{\rho}^{T_2}$. The density matrix $\hat{\rho}$ is inseparable if and only if $\hat{\sigma}$ has any negative eigenvalues [4,15]. The measure of entanglement $E(\hat{\rho})$ is then defined as

$$E(\hat{\rho}) = -2 \sum_i \lambda_i^- \tag{4}$$

where λ_i^- are the negative eigenvalues of $\hat{\sigma}$ and the factor 2 is introduced to have $0 \leq E(\hat{\rho}) \leq 1$. In Appendix, we prove that the entanglement measure (4) satisfies the above necessary conditions.

In fact, there is the fourth condition which a measure of entanglement has to satisfy:

(C.4) For pure states, the measure of entanglement reduces to the entropy of entanglement.

We note that for a pure entangled state the entanglement measure (4) is not reduced to the entropy of entanglement S but is a monotonously increasing function of S as shown in Fig. 2. Vedral and Plenio [14] showed that the Bures metric satisfies the condition (C.1)-(C.3) but is smaller than the entropy of entanglement for pure states. They then wrote that measures which do not satisfy condition (C.4) can nevertheless contain useful information on entanglement. The Schmidt norm is another example of the measure of entanglement which does not satisfy condition (C.4) [16]. The entanglement measure (4) can qualitatively give information on the entanglement of a given state as it satisfies conditions (C.1)-(C.3). Because of convenience in calculation, we use $E(\hat{\rho})$ in Eq. (4) as the measure of entanglement in this paper.

III. PARTIAL TELEPORTATION OF ENTANGLEMENT

We consider the partial teleportation of entanglement as shown in Fig. 1, where Alice teleports one particle of her entangled pair to Bob. Alice and Bob share an ancillary pair of an entangled state. Alice performs the Bell-state measurement on one of her original entangled pair and her part of the ancillary pair. Upon receiving Alice's measurement result through a classical channel, Bob unitary rotates his part of the ancillary pair based on it. If the ancillary pair is perfectly entangled, Bob's particle and Alice's unmeasured particle become entangled as the Alice's original entangled pair. The basic idea of the partial teleportation is similar to the single-particle teleportation or entanglement swapping but our interest here does not stop at a simple result of the teleportation of a particle. In this paper, the quantum channel is represented by a mixed entangled state due to the influence from the environment. We are interested in how the entanglement of the teleported state is affected by such the imperfect quantum channel by studying the channel-dependent fidelity, information transfer and entanglement transfer. We assume for the simplicity that Alice's initial state is pure and the quantum channel is represented by a Werner state [17].

An initial pure state for entangled two particles 1 and 2 is given in the Hilbert-Schmidt space by

$$\hat{\rho}^{12} = \frac{1}{4}\left(\hat{1} \otimes \hat{1} + \vec{a}_0 \cdot \vec{\sigma} \otimes \hat{1} + \hat{1} \otimes \vec{b}_0 \cdot \vec{\sigma} + \sum_{nm} c_0^{nm} \hat{\sigma}_n \otimes \hat{\sigma}_m\right) \tag{5}$$

where $\vec{a}_0$ and $\vec{b}_0$ are real vectors and c_0^{nm} is an element of the real matrix C_0. The initial pure state $\hat{\rho}_0$ in Eq. (5) satisfies the pure state condition $\hat{\rho}^2 = \hat{\rho}$. We can also consider a general representation of the initial pure state (5), with help of the seed state $\hat{\rho}_s^{12}$ defined as follows

$$\hat{\rho}^{12} = (\hat{U}^1 \otimes \hat{U}^2) \hat{\rho}_s^{12} (\hat{U}^1 \otimes \hat{U}^2)^\dagger \tag{6}$$

where $\hat{U}^1$ and $\hat{U}^2$ are local unitary operators acting respectively on the particles 1 and 2. The density operator for the seed state is

$$\hat{\rho}_s^{12} = \frac{1}{4}\left(\hat{1} \otimes \hat{1} + a_0 \hat{\sigma}_z \otimes \hat{1} + \hat{1} \otimes a_0 \hat{\sigma}_z + \sum_n c_n \hat{\sigma}_n \otimes \hat{\sigma}_n\right) \tag{7}$$

where a_0 is a positive real number and $\vec{c} = (c_0, -c_0, 1)$ a real vector, constrained by $a_0^2 + c_0^2 = 1$. The vector $\vec{c}$ describes the quantum correlation of the pure state $\hat{\rho}_s^{12}$, yielding the relation $E_0 = |c_0|$, where E_0 is the measure of entanglement for $\hat{\rho}_s^{12}$. The state has no entanglement, i.e., $E_0 = 0$, if and only if $c_0 = 0$. Now, the state $\hat{\rho}_s^{12}$

is characterised by one parameter c_0 or equivalently by its entanglement E_0. By the definition of the measure of entanglement, the initial state of the density operator $\hat{\rho}^{12}$ and the state of the seed density operator $\hat{\rho}_s^{12}$ have the same measure of entanglement, E_0. It is thus clear that the initial state $\hat{\rho}^{12}$ is fully determined by its measure of entanglement E_0 and the local unitary operations $\hat{U}^1$ and $\hat{U}^2$.

We take the Werner state for the quantum channel. In fact, any mixed state can be made a Werner state by random local $SU(2) \otimes SU(2)$ operations [12,17] so that we do not lose the generality by taking the Werner state in describing the mixed channel. The Werner state $\hat{w}^{34}$ of the ancillary particles 3 and 4 is

$$\hat{w}^{34} = \frac{1}{4}\left(\hat{1} \otimes \hat{1} + \sum_{nm} c_w^{nm} \hat{\sigma}_n \otimes \hat{\sigma}_n\right). \tag{8}$$

where c_w^{nm} is an element of the real matrix $C_w = (2\Phi + 1)/3 \cdot \mathrm{diag}(-1,-1,-1)$. The Werner state becomes the singlet state when $\Phi = 1$. The parameter Φ is related to the entanglement E_w of the Werner state $\hat{w}^{34}$. It is straightforward to show that $E_w = \max(0, \Phi)$.

The Bell-state measurement by Alice is represented by a family of projectors

$$\hat{P}_\alpha^{23} = |\Psi_\alpha^{23}\rangle\langle\Psi_\alpha^{23}| = \frac{1}{4}\left(I \otimes I + \sum_{nm} p_\alpha^{nm} \sigma_n \otimes \sigma_m\right) \tag{9}$$

where $|\Psi_\alpha^{23}\rangle$ are the four possible Bell states and p_α^{nm} is an element of the real matrix P_α; $P_0 = \mathrm{diag}(-1,-1,-1)$, $P_1 = \mathrm{diag}(-1,1,1)$, $P_2 = \mathrm{diag}(1,-1,1)$, $P_3 = \mathrm{diag}(1,1,-1)$. The Bell measurement is performed on the particle 2 of the initial entangled pair and the particle 3 of the Werner state (see Fig. 1).

The quantum teleportation utilises both the classical and quantum channels. Upon receiving the two-bit classical message on the Bell-state measurement through the classical channel, Bob performs the unitary transformation on the particle 4 accordingly. If the quantum channel is in the spin singlet state, the teleportation can be perfectly completed by one of the following four possible unitary operators: $\hat{1}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$. However, for the mixed channel, it is difficult for Bob to decide which unitary transformation to get the final state $\hat{\rho}^{14}$ maximally close to the initial state $\hat{\rho}^{12}$. Let us consider how to determine the right unitary transformation when the channel is in the Werner state.

Suppose Bob receives a two-bit message through the classical channel saying that Alice's measurement was $|\Psi_\alpha^{23}\rangle$. Bob then applies the unitary transformation $\hat{U}_\alpha^4$ on the particle 4, then the state $\hat{\rho}_\alpha^{14}$ of the two particles 1 and 4 becomes

$$\hat{\rho}_\alpha^{14} = \frac{1}{p_\alpha}\mathrm{Tr}_{2,3}\left[\hat{P}_\alpha^{23} \otimes \hat{U}_\alpha^4 \left(\hat{\rho}_0^{12} \otimes \hat{w}^{34}\right) \hat{P}_\alpha^{23} \otimes \hat{U}_\alpha^{4\,\dagger}\right] \tag{10}$$

where p_α is the probability of $|\Psi_\alpha^{23}\rangle$ to be the result of Alice's measurement. Eq. (10) can be written in the Hilbert-Schmidt space as

$$\hat{\rho}_\alpha^{14} = \frac{1}{4}\left(\hat{1} \otimes \hat{1} + \vec{a}_\alpha \cdot \vec{\sigma} \otimes \hat{1} + \hat{1} \otimes \vec{b}_\alpha \cdot \vec{\sigma} + \sum_{nm} \tilde{c}_\alpha^{nm} \hat{\sigma}_n \otimes \hat{\sigma}_m\right). \tag{11}$$

with the parameters

$$\begin{aligned}
\vec{a}_\alpha &= \vec{a}_0, \\
\vec{b}_\alpha &= \frac{2\Phi+1}{3} O_\alpha^T P_\alpha \vec{b}_0, \\
C_\alpha &= \frac{2\Phi+1}{3} O_\alpha^T P_\alpha C_0.
\end{aligned} \tag{12}$$

Here we have the rotation matrix O_α in the Bloch space for a single particle state, obtained from the unitary operator $\hat{U}_\alpha$ [4]:

$$\hat{U}_\alpha \vec{a} \cdot \vec{\sigma} \hat{U}_\alpha^\dagger = \left(O_\alpha^T \vec{a}\right) \cdot \vec{\sigma}. \tag{13}$$

The fidelity F measures how close the final state $\hat{\rho}^{14}$ is to the initial state $\hat{\rho}^{12}$; $F = \sum_\alpha p_\alpha \mathrm{Tr} \hat{\rho}^{12} \hat{\rho}_\alpha^{14}$. If the teleportation is perfect, the final state is the same as the initial state so that the fidelity is 1. By substituting $\hat{\rho}^{12}$ in (5) and $\hat{\rho}_\alpha^{14}$ in (12) into the definition of the fidelity, we find the fidelity for the Werner-state channel

$$\begin{aligned}
F = \frac{1}{4}\Bigg[1 + |\vec{a}_0|^2 &+ \frac{2\Phi+1}{3} \vec{b}_0 \cdot \left(-\frac{1}{4}\sum_\alpha O_\alpha^T P_\alpha \vec{b}_0\right) \\
&+ \frac{2\Phi+1}{3} \mathrm{Tr}\left(-\frac{1}{4}\sum_\alpha O_\alpha^T P_\alpha C_0^T C_0\right)\Bigg].
\end{aligned} \tag{14}$$

The task is now to find Bob's unitary operations $\hat{U}_\alpha$ to maximise fidelity (14). For a general mixed channel, the fidelity is a function of the initial and channel states as well as Bob's unitary operation. However, the basic assumption of the quantum teleportation is that the initial state is unknown. In order to examine the faithfulness of quantum teleportation, we need to average the fidelity over the Hilbert space where the initial state lies in. The unitary operations should be determined to maximise the average fidelity [4].

For the Werner-state channel the fidelity has been calculated as in Eq. (14), where the measurement dependence is found in the terms including $-\sum_\alpha O_\alpha^T P_\alpha$. It is clear that $-P_\alpha$ in Eq. (9) is a rotation matrix thus $|O_\alpha^T P_\alpha| \leq 1$. Choosing $O_\alpha = -P_\alpha$ to maximise the fidelity (14), we find that the corresponding unitary operators are the same as in the singlet-state channel discussed above. This choice enables Bob to produce the measurement-independent final state $\hat{\rho}_\alpha^{14} = \hat{\rho}^{14}$.

The fidelity for the Werner-state channel is then given by

$$F=\frac{1}{4}\left(1+|\vec{a}_0|^2+\frac{2\Phi+1}{3}|\vec{b}_0|^2+\frac{2\Phi+1}{3}\mathrm{Tr}C_0^TC_0\right). \tag{15}$$

It is seen that the fidelity is invariant for the local unitary transformations on the initial pure state. Noting $|\vec{a}_0|$, $|\vec{b}_0|$ and $\mathrm{Tr}C_0^TC_0$ are uniquely determined by the entanglement E_0 for the initial state, the fidelity can be finally written as

$$F=\frac{E_w+2}{3}+\frac{E_w-1}{6}E_0^2. \tag{16}$$

It is clear that the fidelity (16) depends on the initial-state entanglement E_0 and channel entanglement E_w.

The large entanglement in the channel enhances the fidelity and the maximally entangled channel gives the unit fidelity independent from the initial-state entanglement E_0. When Alice's initial state is disentangled, *i.e.* $E_0=0$, it can be written as a direct product of two individual states and the partial entanglement teleportation becomes equivalent to a single-particle case. In this case, the fidelity is $F=(E_w+2)/3$, equal to that for the single-particle teleportation [4]. It has an upper bound of 2/3 producible by classical communication of $E_w=0$ [3]. For the entangled initial state with $E_0\neq 0$, the large initial-state entanglement E_0 reduces monotonously the fidelity (16) for a given channel entanglement $E_w<1$ because the sign of the second term is negative in Eq. (16). This implies that the initial entanglement has fragile nature to teleport. In other words, the entanglement is destroyed easily by the environment.

To examine the loss of the initial entanglement, we consider the entanglement transfer by teleportation. We are interested in how much the initial-state entanglement is transferred to the final state. The measure of entanglement for the final state is calculated as

$$E(\hat{\rho}^{14})=\frac{1}{3}\left[\sqrt{(1-E_w)^2+3E_w(2+E_w)E_0^2}-(1-E_w)\right]. \tag{17}$$

As $\partial E/\partial E_w\geq 0$, large entanglement of the channel enhances the entanglement transfer to the final state from the initial one. For a given entanglement of the channel, the entanglement of the final state increases as the initial-state entanglement gets larger. The entanglement of the final state is nonzero as far as $E_w\neq 0$ and $E_0\neq 0$, which shows that the entangled channel transfers at least some of the initial entanglement to the final state.

IV. CORRELATION INFORMATION

Brukner and Zeilinger have recently derived a measure of information for a quantum state [18]. The information measure is invariant for a choice of a complete set $\{\hat{A}_1,\cdots,\hat{A}_m\}$ of complementary observations and is conserved as far as there is no information exchange between the system and the environment [18]. We employ the measure of information to study the quantum information transfer in the partial teleportation of entanglement.

Suppose an experimental arrangement for a measurement by observable $\hat{A}_j$ which has n possible outcomes with n dimensional probability vector $\vec{p}=(p_1,...,p_i,...,p_n)$ for a given system. The system is supposed to have maximum k-bits of information such that $n=2^k$. The measure of information I_j for the observable $\hat{A}_j$ is defined as

$$I_j=\mathcal{N}\sum_{i=1}^{n}\left(p_i-\frac{1}{n}\right)^2. \tag{18}$$

where the normalisation constant $\mathcal{N}=2^kk/(2^k-1)$. I_j results in k bits of information if one $p_i=1$ and 0 bits of information if all p_i are equal. For a complete set of m mutually complementary observables the measure of information is defined as the sum of the measures of information over the complete set

$$I(\hat{\rho})=\sum_{j=1}^{m}I_j. \tag{19}$$

for the quantum state $\hat{\rho}$. A single spin-1/2 system, for example, is represented by the measure of information $I(\hat{\rho})=2\mathrm{Tr}\hat{\rho}^2-1$.

In the following, we define the measure of correlation information based on the measure of information introduced by Brukner and Zeilinger. Let us consider the measure of information for a composite system of two particles which can be decomposed into three parts. Each particle has its own information corresponding to its reduced density matrix, which we call the *individual information.* The two particles can also have the correlation information which depends on how much they are correlated.

The measure of total information for a density matrix $\hat{\rho}$ of the two spin-1/2 particles is [18]

$$I(\hat{\rho})=\frac{2}{3}\left(4\mathrm{Tr}\hat{\rho}^2-1\right). \tag{20}$$

The measures of individual information $I_a(\hat{\rho})$ and $I_b(\hat{\rho})$ for the particles a and b are

$$I_a(\hat{\rho})=2\mathrm{Tr}_a\left(\hat{\rho}_a\right)^2-1 \tag{21}$$
$$I_b(\hat{\rho})=2\mathrm{Tr}_b\left(\hat{\rho}_b\right)^2-1 \tag{22}$$

where $\hat{\rho}_{a,b}=\mathrm{Tr}_{b,a}\hat{\rho}$ are reduced density matrices for the particles a and b. If the total density matrix $\hat{\rho}$ is represented by $\hat{\rho}=\hat{\rho}_a\otimes\hat{\rho}_b$, the total system is completely separable and we know that there is no correlation information. We thus define the measure of correlation information as

$$I_c(\hat{\rho}) = I(\hat{\rho}) - I(\hat{\rho}_a \otimes \hat{\rho}_b) = I(\hat{\rho}) - \frac{2}{3}[I_a(\hat{\rho}) + I_b(\hat{\rho}) + I_a(\hat{\rho})I_b(\hat{\rho})]. \quad (23)$$

The measures of individual and correlation information are invariant for any particular choice of the complete set of complementary observables.

If there is no correlation between the two particles, the measure of total information is a mere sum of the measures of individual information. On the other hand, the total information is imposed only on the correlation information, $I = I_c$, if there is no individual information, $I_a = I_b = 0$. For a two spin-1/2 system, the maximally entangled states have only the correlation information.

Note that the correlation information is not the same as the measure of entanglement. Only when a pure state is considered the measure of the correlation is directly related to the measure of entanglement. For a mixed state, the correlation information also includes the information due to classical correlation. For example, the state of the density operator $\hat{\rho}_{cc} = 1/4(\hat{1}\otimes\hat{1} - \sigma_z \otimes \sigma_z)$ is not quantum-mechanically entangled but classically correlated with the correlated information $I_c \neq 0$ [19].

The total information $I(\hat{\rho}^{14})$ in the final state (11) is obtained using its definition (20):

$$I(\hat{\rho}^{14}) = \frac{2}{3}\left[1 + 2\left(\frac{2E_w+1}{3}\right)^2 + \left\{\left(\frac{2E_w+1}{3}\right)^2 - 1\right\}E_0^2\right] \quad (24)$$

which depends on the initial-state and the channel entanglement. We can easily find that $0 \leq I(\hat{\rho}^{14}) \leq 2$ from the range of the entanglement measure $0 \leq E_w,\ E_0 \leq 1$. The measure of information for the initial state is 2 as it is a pure two spin-1/2 system. The final state can have at best the same amount of information as the initial state because the noisy environment acts only to dissipate the information. The total information is better preserved for the larger channel entanglement E_w. The total information is lost more easily for the larger initial entanglement as the sign of the coefficient for E_0^2 is negative. This is consistent with the discussions for the fidelity.

We evaluate the measures of individual I_1, I_4 and correlation I_c information for the final state given by

$$I_1(\hat{\rho}^{14}) = I_1^0 \quad (25)$$

$$I_4(\hat{\rho}^{14}) = \left(\frac{2E_w+1}{3}\right)^2 I_2^0 \quad (26)$$

$$I_c(\hat{\rho}^{14}) = \left(\frac{2E_w+1}{3}\right)^2 I_c^0 \quad (27)$$

where $I_1^0 = 1 - E_0^2$, $I_2^0 = 1 - E_0^2$, and $I_c^0 = 2(4 - E_0^2)E_0^2/3$ are the measures of individual and correlation information for the initial state $\hat{\rho}^{12}$. Because there has been no action on Alice's particle 1, its individual information remains unchanged with $I_1 = I_1^0$. On the other hand, the individual information for the particle 2 is not fully transferred to the particle 4 and the correlation information is decreased. The coefficient for the decrease of the information is the same for I_4 and I_c but we must remember that the maximum measure of correlation information is 2 while that of individual information is 1, which shows that the correlation information can be lost more easily.

Because the final state is a mixed state, its correlation information I_c describes in general both the quantum and classical correlations. For an entangled initial state of $E_0 \neq 0$ we consider two cases: $E_w = 0$ and $E_w \neq 0$. If $E_w = 0$, the final state is classically correlated because $I_c \neq 0$ in Eq. (27) whereas $E = E(\hat{\rho}^{14}) = 0$ in Eq. (17). On the other hand, if $E_w \neq 0$, the correlation information I_c can be written in terms of the final entanglement E:

$$I_c = 2\left(\frac{2E_w+1}{3}\right)^2\left(4 - 3\frac{E + (1-E_w)}{E_w(2+E_w)}E\right) \times \frac{E + (1-E_w)}{E_w(2+E_w)}E. \quad (28)$$

The correlation information $I_c = 0$ if and only if $E = 0$ for partial teleportation via the Werner channel. This shows that for $E_w \neq 0$ the correlation information of the final state is only due to the quantum correlation.

V. REMARKS

The partial teleportation of entanglement has been considered in the noisy environment. The measures of individual and correlation information have been extensively studied for the two spin-1/2 system. As the Werner-state is employed for the quantum channel, the calculation of the fidelity, the information transfer and the entanglement transfer became extremely simple while we do not lose the generality to consider the noisy environment. The larger the initial-state entanglement is, the worse the fidelity becomes for any imperfect quantum channel. We, however, find more entanglement in the final state with the larger initial-state entanglement. The entangled channel transfers at least some of the entanglement to the final state.

ACKNOWLEDGMENTS

JL thanks Mr.Č. Bruckner for discussions at the Entanglement and Decoherence Workshop in Garda. This work was supported by the Brain Korea 21 grant (D-0055) of the Korean Ministry of Education.

APPENDIX: PROOF OF THE MEASURE OF ENTANGLEMENT EQ. (4)

Consider a density matrix $\hat{\rho}$ for a two spin-1/2 system and the partial transposition $\hat{\sigma} = \hat{\rho}^{T_2}$ [20]. The density matrix $\hat{\rho}$ is inseparable if and only if $\hat{\sigma}$ has any negative eigenvalues [4,15]. The measure of entanglement $\mathcal{E}(\hat{\rho})$ is defined as $2\sum_i(-\lambda_i^-)$ with the negative eigenvalues λ_i^- of $\hat{\sigma}$. We will show that $\mathcal{E}(\hat{\rho})$ satisfies the three conditions (C.1)-(C.3).

Let $\hat{d}(\hat{\rho})$ be a diagonal matrix of the partial transposition $\hat{\sigma}$ such that, for some unitary operator U,

$$\begin{aligned} \hat{d}(\hat{\rho}) &= \left(U\hat{\sigma}U^\dagger\right) \\ &= \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \end{aligned} \tag{A1}$$

where $\mathrm{diag}(\{\lambda_i\})$ represents a diagonal matrix with its diagonal elements λ_i and thus λ_i are eigenvalues of $\hat{\sigma}$. For the given $\hat{d}(\hat{\rho})$, the density matrix space is decomposed into two subspaces; one is expanded by eigenvectors of the semi-positive (positive or zero) eigenvalues and the other of the negative eigenvalues of $\hat{d}$. The identity operator $\hat{1}$ is then the sum of two projectors $\hat{P}_+$ and $\hat{P}_-$ such that

$$\hat{1} = \hat{P}_+ + \hat{P}_- \tag{A2}$$

where $\hat{P}_+$ ($\hat{P}_-$) projects the density matrix space onto the semi-positive (negative) eigenvalue subspace. Any hermitian matrix $\hat{H}$ is decomposed into

$$\hat{H} = \hat{P}_+\hat{H}\hat{P}_+ + \hat{P}_+\hat{H}\hat{P}_- + \hat{P}_-\hat{H}\hat{P}_+ + \hat{P}_-\hat{H}\hat{P}_- \tag{A3}$$

and thus

$$\hat{d}(\hat{\rho}) = \hat{d}_+(\hat{\rho}) + \hat{d}_-(\hat{\rho}) \tag{A4}$$

where $\hat{d}_+ \equiv \hat{P}_+\hat{d}\hat{P}_+$ and $\hat{d}_- \equiv \hat{P}_-\hat{d}\hat{P}_-$. Note that $\hat{P}_+\hat{d}\hat{P}_- = \hat{P}_-\hat{d}\hat{P}_+ = 0$ since $\hat{d}$ is a diagonal matrix. Now, the measure of entanglement $\mathcal{E}(\hat{\rho})$ is defined as twice the absolute value of the trace on the negative diagonal matrix $\hat{d}_-(\hat{\rho})$, given by

$$\mathcal{E}(\hat{\rho}) \equiv -2\mathrm{Tr}\left[\hat{d}_-(\hat{\rho})\right] = 2\sum_\beta(-\lambda_\beta^-) \tag{A5}$$

where λ_β^- is negative eigenvalue of $\hat{\sigma}$ and the factor 2 is introduced to be $0 \le \mathcal{E}(\hat{\rho}) \le 1$.

Now we consider that $E(\hat{\rho})$ in Eq. (4) satisfies the necessary conditions (C.1)-(C.3). If $\hat{\rho}$ is separable, $\hat{\sigma}$ has no negative eigenvalues and the converse statement also holds, satisfying the condition (C.1). A local unitary transformation leads to new density matrix $\hat{\rho}' = \hat{U}_1 \otimes \hat{U}_2\hat{\rho}\hat{U}_1^\dagger \otimes \hat{U}_2^\dagger$ and its partial transposition $\hat{\sigma}' = \hat{U}_1 \otimes \hat{U}_2^*\hat{\sigma}\hat{U}_1^\dagger \otimes (\hat{U}_2^*)^\dagger$. Note that $\hat{U}_1 \otimes \hat{U}_2^*$ is a unitary operator such that $\hat{U}_1 \otimes \hat{U}_2^*\hat{U}_1^\dagger \otimes (\hat{U}_2^*)^\dagger = \hat{1}$. Since the eigenvalues are independent of the unitary transformation, the condition (C.2) is satisfied with $E(\hat{\rho}') = E(\hat{\rho})$.

To consider the final condition (C.3), we introduce the LGM+CC(+PSS) which maps the density matrix $\hat{\rho}$ into $\hat{\rho}'$, defined by

$$\hat{\rho}' = \sum_i \hat{V}_i\hat{\rho}\hat{V}_i^\dagger \tag{A6}$$

where the classically correlated operator $\hat{V}_i = \hat{A}_i \otimes \hat{B}_i$ satisfies the complete relation as $\sum_i(\hat{V}_i)^\dagger\hat{V}_i = \hat{1}$. Let $p_i\hat{\rho}_i = \hat{V}_i\hat{\rho}\hat{V}_i^\dagger$ with $p_i = \mathrm{Tr}\hat{V}_i\hat{\rho}\hat{V}_i^\dagger$ and $\hat{\sigma}_i = \hat{\rho}_i^{T_2}$. Since $\hat{V}_i$ is a local operator, $\hat{\sigma}_i$ is represented in terms of $\hat{\sigma}$, namely,

$$p_i\hat{\sigma}_i = \hat{V}_i'\hat{\sigma}\ \hat{V}_i'^{\,\dagger} \tag{A7}$$

where $\hat{V}_i' = \hat{A}_i \otimes \hat{B}_i^*$ is an LGM+CC operator with a completeness relation $\sum_i \hat{V}_i'^{\,\dagger}\,\hat{V}_i' = \hat{1}$.

Suppose $\hat{d}^i$ are diagonal matrices of $\hat{\sigma}_i$ with some unitary operator $\hat{U}_i$ and $\hat{d}$ of $\hat{\sigma}$ with $\hat{U}$. The diagonal matrix $\hat{d}^i$ can be written as

$$\begin{aligned} p_i\hat{d}^i &= \left(\hat{U}_i\hat{V}_i'\hat{U}^\dagger\right)\left(\hat{U}\hat{\sigma}\hat{U}^\dagger\right)\left(\hat{U}\ \hat{V}_i'^{\,\dagger}\,\hat{U}_i^\dagger\right) \\ &= \hat{W}_i\hat{d}\hat{W}_i^\dagger \end{aligned} \tag{A8}$$

where $\hat{W}_i = \hat{U}_i\hat{V}_i'\hat{U}_0^\dagger$ is also a LGM+CC operator. For the given $\hat{d}^i$, two projectors $\hat{P}_-^i$ and $\hat{P}_+^i$ are defined to project the density matrix space onto semi-positive and negative eigenvalue subspaces of $\hat{d}^i$. The measure of entanglement $E(\hat{\rho}_i)$ on the subensemble $\hat{\rho}_i$ is given by

$$\begin{aligned} p_iE(\hat{\rho}_i) &= -2p_i\mathrm{Tr}\left[\hat{d}_-^i(\hat{\rho}_i)\right] \\ &= -2\mathrm{Tr}\left(\hat{P}_-^i\hat{W}_i\hat{d}\hat{W}_i^\dagger\hat{P}_-^i\right). \end{aligned} \tag{A9}$$

We represent $\hat{d} = \sum_j \lambda_j|\psi_j\rangle\langle\psi_j| = \sum_\alpha \lambda_\alpha^+|\psi_\alpha\rangle\langle\psi_\alpha| + \sum_\beta \lambda_\beta^-|\psi_\beta\rangle\langle\psi_\beta|$ where the sum is decomposed into two sums of semi-positive and negative eigenvalues. The whole-ensemble average of the measures of entanglement is given by

$$\begin{aligned} \sum_i p_iE(\hat{\rho}_i) &= 2\sum_{ij}(-\lambda_j)\langle\psi_j|\hat{W}_i^\dagger\hat{P}_-^i\hat{P}_-^i\hat{W}_i|\psi_j\rangle \\ &= 2\sum_{i\alpha}(-\lambda_\alpha^+)\langle\psi_\alpha|\hat{W}_i^\dagger\hat{P}_-^i\hat{P}_-^i\hat{W}_i|\psi_\alpha\rangle \\ &\quad +2\sum_{i\beta}(-\lambda_\beta^-)\langle\psi_\beta|\hat{W}_i^\dagger\hat{P}_-^i\hat{P}_-^i\hat{W}_i|\psi_\beta\rangle \end{aligned} \tag{A10}$$

where we separate the sum into the sums of semi-positive eigenvalues λ_α^+ and negative eigenvalues λ_β^- of $\hat{d}$. The inequality, $\langle\psi_j|\hat{W}_i^\dagger\hat{P}_-^i\hat{P}_-^i\hat{W}_i|\psi_j\rangle \ge 0$, and the sign of eigenvalues make the first term negative and the second term

positive in Eq. (A10). Eq. (A10) results in the following inequality

$$\sum_i p_i E(\hat{\rho}_i) \le 2\sum_{i\beta}(-\lambda_\beta^-)\langle\psi_\beta|\hat{W}_i^\dagger \hat{P}_-^i \hat{P}_-^i \hat{W}_i|\psi_\beta\rangle. \quad \text{(A11)}$$

Since $0 \le \sum_i \langle\psi|\hat{W}_i^\dagger \hat{P}_-^i \hat{P}_-^i \hat{W}_i|\psi\rangle \le 1$ for arbitrary wave function $|\psi\rangle$ [21], we finally arrive at the inequality

$$\begin{aligned}\sum_i p_i E(\hat{\rho}_i) &\le 2\sum_\beta(-\lambda_\beta^-)\sum_i\langle\psi_\beta|\hat{W}_i^\dagger \hat{P}_-^i \hat{P}_-^i \hat{W}_i|\psi_\beta\rangle \\ &\le 2\sum_\beta(-\lambda_\beta^-) = E(\hat{\rho}), \end{aligned}\quad \text{(A12)}$$

which satisfies the condition (C.3) for an arbitrary set of LGM+CC operators.

As an example how to calculate the measure of entanglement, consider the Werner state $\hat{w}$ in Eq. (8). When we select the following representations,

$$\hat{1}\otimes\hat{1} = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \qquad \hat{\sigma}_x\otimes\hat{\sigma}_x = \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},$$
$$\hat{\sigma}_y\otimes\hat{\sigma}_y = \begin{pmatrix} 0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}, \quad \hat{\sigma}_z\otimes\hat{\sigma}_z = \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}, \quad \text{(A13)}$$

the Werner state is written as

$$\hat{w} = \begin{pmatrix} \frac{1-f}{4} & 0 & 0 & 0\\ 0 & \frac{1+f}{4} & -\frac{f}{2} & 0\\ 0 & -\frac{f}{2} & \frac{1+f}{4} & 0\\ 0&0&0&\frac{1-f}{4}\end{pmatrix} \quad \text{(A14)}$$

where $f = (2\Phi+1)/3$, and the set of eigenvalues is $\{(1-f)/4, (1-f)/4, (1-f)/4, (1+3f)/4\}$. The positivity of density matrix requires that $-1/3 \le f \le 1$ and thus $-1 \le \Phi \le 1$. The partial transposition is now given by

$$\hat{\sigma} = \hat{w}^{T_2} = \begin{pmatrix} \frac{1-f}{4} & 0 & 0 & -\frac{f}{2}\\ 0 & \frac{1+f}{4} & 0 & 0\\ 0 & 0 & \frac{1+f}{4} & 0\\ -\frac{f}{2}&0&0&\frac{1-f}{4}\end{pmatrix} \quad \text{(A15)}$$

which has its eigenvalues $\{(1+f)/4, (1+f)/4, (1+f)/4, (1-3f)/4\}$. It is clear that three eigenvalues are positive since $(1+f)/4 \ge 0$ under the constraint of $-1/3 \le f \le 1$. The other eigenvalue can be negative only if $3f > 1$ or equivalently $\Phi > 0$. For $3f > 1$, $E(\hat{\rho}) \equiv 2\sum_\beta(-\lambda_\beta^-) = (3f-1)/2$ while $E(\hat{\rho}) = 0$ for $3f \le 1$. The Werner state with $f = 1$ becomes a maximally entangled singlet state and then $E(\hat{w}) = 1$.

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[19] The density operator for the classically-correlated state $\hat{\rho}_{cc}$ is written in more conventional representation as $\hat{\rho} = (1/2)(|\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow|)$.

[20] When a density matrix has the components ρ_{ijkl} in an orthonormal basis set $\{|e_i^a\rangle\langle e_j^a|\otimes|e_k^b\rangle\langle e_l^b|\}$, its partial transposition has the components obtained as $\sigma_{ijkl} = \rho_{ijlk}$.

[21] Because of the completeness relation of LGM operators $\sum_i \hat{W}_i^\dagger \hat{W}_i = \hat{1}$, the following identity holds

$$\begin{aligned}1 &= \sum_i\langle\psi|\hat{W}_i^\dagger\hat{W}_i|\psi\rangle \\ &= \sum_i\langle\psi|\hat{W}_i^\dagger\hat{P}_+^i\hat{W}_i|\psi\rangle + \sum_i\langle\psi|\hat{W}_i^\dagger\hat{P}_-^i\hat{W}_i|\psi\rangle\end{aligned}$$

for any wave function $|\psi\rangle$ and because $\langle\psi|\hat{W}_i^\dagger\hat{P}_\pm^i\hat{W}_i|\psi\rangle = |\hat{P}_\pm^i\hat{W}_i|\psi\rangle|^2 \ge 0$,

$$0 \le \sum_i\langle\psi|\hat{W}_i^\dagger\hat{P}_\pm^i\hat{W}_i|\psi\rangle \le 1.$$

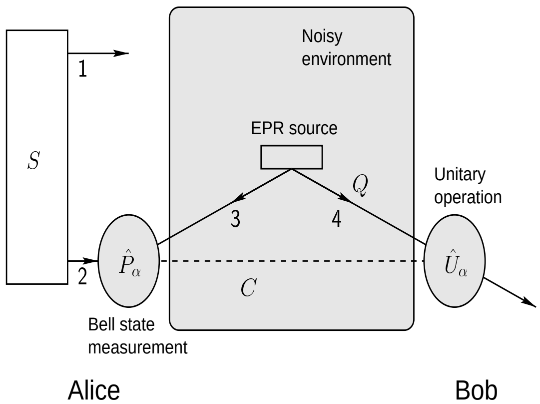

FIG. 1. Schematic drawing for the partial teleportation. An unknown spin-1/2 two-body quantum entangled state is generated by Alice's source S. The quantum channel is produced and Alice and Bob share the correlated pair. Alice performs the Bell measurement on the particles 2 and 4 and sends the result to Bob through the classical channel. Bob unitarily transforms the particle 4 to complete the partial teleportation. We are interested in the entanglement and closeness of the state of particles 1 and 4 to the initial state of particles 1 and 2.

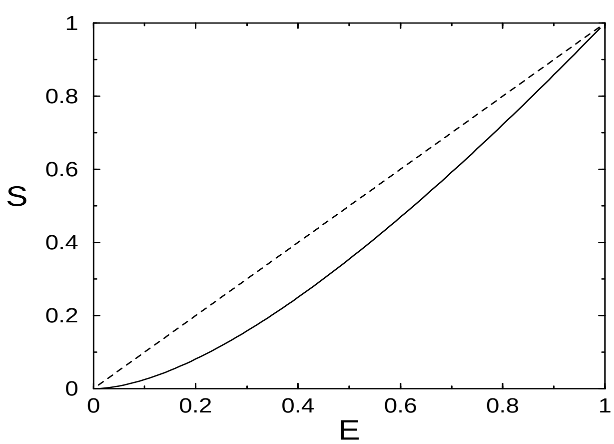

FIG. 2. The entropy of entanglement S (solid line) in terms of the measure of entanglement E for a pure spin-1/2 entangled state. The dashed line is an eye-guidance for a linear curve. The entropy of entanglement S is a monotonously increasing function of the measure of entanglement E for a pure state because the first derivative of S is positive everywhere from 0 to 1 of E.